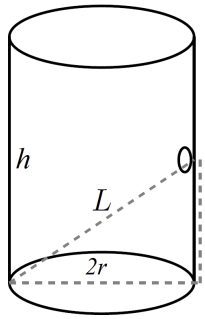
MATH 102:107, CLASS 16 (FRI OCT 13)

(1) (Constrained optimization - Kepler's wedding) A cylindrical wine barrel has a hole in the center of one side. When a rod is put into this hole and reaches the furthest into the barrel that it can go, it reaches a distance of L. Given this constraint, find the radius r and height h which maximize the volume of the barrel.



Solution: Let r denote the radius of the barrel and let h denote the height.

$$\text{Constraint}: \quad (2r)^2 + (h/2)^2 = L^2$$

$$\text{Objective function}: \quad \pi r^2 h$$

We isolate r in the constraint equation:

$$4r^2 + h^2/4 = L^2 \implies r = \pm\sqrt{\frac{L^2}{4} - \frac{h^2}{16}}$$

and plug this expression for r back into the objective function to get it as a function of just the variable h.

$$f(h) = \pi r^2 h = \pi\left(\frac{L^2}{4} - \frac{h^2}{16}\right)h$$

$$f(h) = \frac{\pi L^2}{4}h - \frac{\pi}{16}h^3$$

In the situation of the problem, $h > 0$ (because height must be positive) and $h < 2L$ (because L is at least half the height) - therefore, we are looking for the global maximum of $f(h)$ in the interval $(0, 2L)$. We first find the critical points:

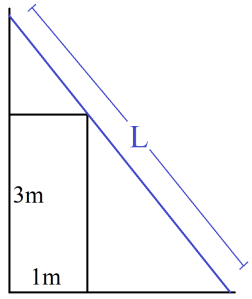
$$f'(h) = \frac{\pi L^2}{4} - \frac{3\pi}{16}h^2 = 0$$

$$\implies h^2 = \frac{4}{3}L^2 \implies \boxed{h = \frac{2}{\sqrt{3}}L}$$

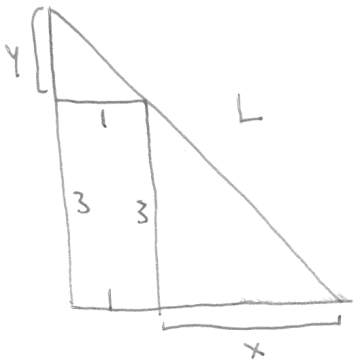
This is a local maximum because $f''(h) = -\frac{3\pi}{8}h$ which is negative at $h = \frac{2}{\sqrt{3}}L$. And it is a global maximum, because it's the only critical point. We can even check the endpoints of the interval:

$$f(0) = 0 \qquad f\left(\frac{2}{\sqrt{3}}L\right) = \frac{\pi}{3\sqrt{3}}L^3 \qquad f(2L) = 0$$

(2) (Constrained optimization) A box of height 1m and depth 3m is placed against a wall. A straight ladder must go over the box and lean against the wall. What is the shortest possible length of the ladder?



Solution 1: Constrained optimization.



$L^2 = (x+1)^2 + (y+3)^2 \leftarrow$ Objective function

$\frac{y}{1} = \frac{3}{x} \longleftarrow$ Constraint

Variables: x, y

Let x be the distance from the foot of the ladder to the box, and let y be the distance from the top of the box to the top of the ladder. Then our objective function, which we want to minimize, is[1]

$$L^2 = (x+1)^2 + (y+3)^2$$

x and y can't vary freely, though - they are related. The constraint is that the ladder is a straight line - i.e., the right triangle formed by x and the height of the box, is similar to the right triangle formed by the width of the box and y. These similar triangles give us the constraint equation

$$y = \frac{3}{x}$$

[1]Minimizing L is the same as minimizing L^2, so we'll just do the second because it's easier.

Plugging this into the objective function gives us

$$f(x) = L^2 = (x+1)^2 + \left(\frac{3}{x} + 3\right)^2$$

$$= (x+1)^2 + \left(\frac{3}{x}(x+1)\right)^2 = (x+1)^2\left(1 + \frac{9}{x^2}\right)$$

Take the derivative:

$$f'(x) = 2(x+1)\left(1 + \frac{9}{x^2}\right) + (x+1)^2\left(-\frac{18}{x^3}\right)$$

$$= 2(x+1)\left(1 + \frac{9}{x^2} - (x+1)\frac{9}{x^3}\right)$$

$$= 2(x+1)\left(1 - \frac{9}{x^3}\right)$$

Remember that $x > 0$, because the foot of the ladder has to be to the right of the box. Therefore, the only critical point is $\boxed{x = \sqrt[3]{9}}$. To check that this is indeed a minimum, we have to take the second derivative

$$f''(x) = 2\left(1 - \frac{9}{x^3}\right) + 2(x+1)\frac{27}{x^4} = 2 + \frac{36}{x^3} + \frac{54}{x^4}$$

which is positive at $x = \sqrt[3]{9}$. Therefore, $f(x)$ is *concave up* at $x = \sqrt[3]{9}$, meaning that $x = \sqrt[3]{9}$ is indeed a local minimum.

To find the length of the ladder in this case, we can solve for y: $y = \frac{3}{\sqrt[3]{9}} = \sqrt[3]{3}$. Then we can use the Pythagorean theorem

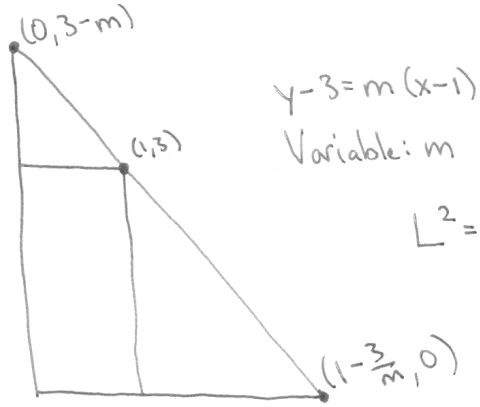
$$L^2 = (\sqrt[3]{9} + 1)^2 + (\sqrt[3]{3} + 3)^2$$

$$= \sqrt[3]{81} + 2\sqrt[3]{9} + 1 + \sqrt[3]{9} + 6\sqrt[3]{3} + 9$$

$$= 3\sqrt[3]{9} + 9\sqrt[3]{3} + 10$$

and so $L = \sqrt{3\sqrt[3]{9} + 9\sqrt[3]{3} + 10}$.

Solution 2: Unconstrained optimization.



$y - 3 = m(x-1)$

Variable: m (slope)

$L^2 = (1 - \frac{3}{m})^2 + (3-m)^2$ ← Objective function

We can put the picture in a coordinate system. The upper-right corner of the box is at the point $(1, 3)$, and the ladder is a line through this point with some (negative) slope m. Thus, the equation of the ladder is $y - 3 = m(x - 1)$. We can calculate the intersections of this line with the x-axis

$$-3 = m(x-1) \implies x = 1 - \frac{3}{m}$$

and y-axis

$$y - 3 = m(-1) \implies y = 3 - m$$

Our objective function is thus

$$f(m) = L^2 = \left(1 - \frac{3}{m}\right)^2 + (3-m)^2$$

which factors

$$\left(\frac{m-3}{m}\right)^2 + (m-3)^2 = (m-3)^2\left(1 + \frac{1}{m^2}\right)$$

To minimize, we calculate the derivative

$$f'(m) = 2(m-3)\left(1 + \frac{1}{m^2}\right) + (m-3)^2\left(\frac{-2}{m^3}\right)$$

$$= 2(m-3)\left(1 + \frac{1}{m^2} - \frac{1}{m^2} + \frac{3}{m^3}\right) = 2(m-3)\left(1 + \frac{3}{m^3}\right)$$

Remember that m must be negative, because the foot of the ladder must be to the right of the box. Therefore, the only critical point is $\boxed{m = -\sqrt[3]{3}}$. To check this is a local minimum of $f(m)$, we can calculate the second derivative $f''(m)$ - this is similar to in Solution 1. We can also find the length of the ladder for this value of m, and will get the same answer as in Solution 1.

(3) (Constrained optimization) *Baculovirus* is a cylindrically-shaped cell which must hold a certain amount of genetic material, and therefore has fixed volume 54000π nm^3. Find the radius and height which give the cell the minimal possible surface area.

Solution: Our constraint is $\pi r^2 h = 54000\pi$, or equivalently, $r^2 h = 54000$. We must minimize the surface area, which is $2\pi r^2 + 2\pi r h$. We can isolate h in the constraint equation

$$h = \frac{54000}{r^2}$$

and plug it into the objective function to get

$$f(r) = 2\pi r^2 + 2\pi r \frac{54000}{r^2} = 2\pi r^2 + 2\pi \frac{54000}{r}$$

We are looking for a global minimum of this function on the interval $(0, \infty)$. Finding the critical points:

$$f'(r) = 2\pi\left(2r - \frac{54000}{r^2}\right) = 0$$

$$\implies r - \frac{27000}{r^2} = 0 \implies \boxed{r = 30}$$

This is a local minimum because $f''(r) = 2\pi(2 + \frac{108000}{r^3})$ is positive at $r = 30$. It's the only critical point, and therefore is a global minimum. (Checking the endpoints of the interval: as $r \to 0$, $f(r) \to \infty$ and as $r \to \infty$, $f(r) \to \infty$.)

(4) (Unconstrained optimization) Let x measure the population of aphids in a garden. The reproduction rate of aphids is $G(x) = 3x$ and the rate of predation by ladybugs is $P(x) = \frac{30x}{5+x}$. Is there a value of $x > 0$ for which the net growth rate is *minimized?* At which it is *maximized?* For each, either find the value of x, or explain why none exists.

Solution: The net growth rate is

$$N(x) = G(x) - P(x) = 3x - \frac{30x}{5+x}$$

where x varies over the interval $[0, \infty)$. First, let's calculate the critical points of $N(x)$.

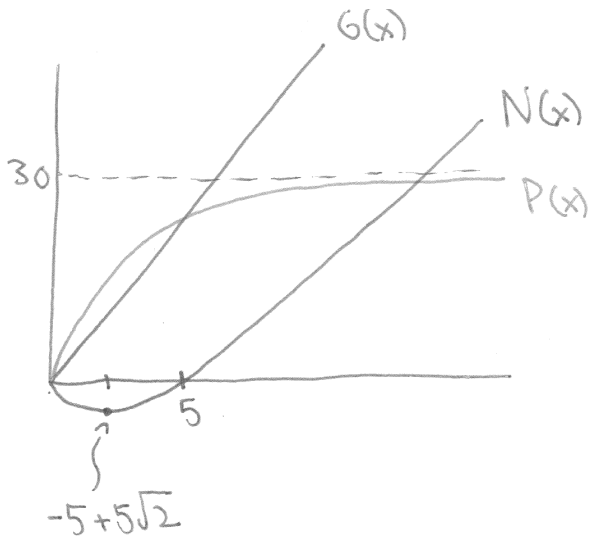
$$N'(x) = 3 - \frac{150}{(5+x)^2} = 0 \iff 3(5+x)^2 = 150$$

$$\iff 5 + x = \pm 5\sqrt{2} \iff x = -5 \pm 5\sqrt{2}$$

We only need to consider the positive root, as the negative root lies outside of the range of the model. To classify this critical point, we calculate the second derivative

$$N''(x) = \frac{300}{(5+x)^3} \implies N(-5+5\sqrt{2}) = \frac{300}{5^3} = \frac{12}{5}$$

which is positive - therefore, $x = -5 + 5\sqrt{2}$ is a local *minimum.* The derivative $N'(x) = 3 - \frac{150}{(5+x)^2}$ is negative for $0 < x < -5 + 5\sqrt{2}$, and positive for all $x > -5 + 5\sqrt{2}$. Therefore, $-5 + 5\sqrt{2}$ is a *global minimum.* The function has no global maximum, because as $x \to \infty$, $N(x) \to \infty$ (because $\frac{30x}{5+x}$ approaches towards the maximal rate of 30, while $3x$ grows without bound).



(5) (General problem solving) Let $f(x) = 4 - x^2$. Calculate the equation of the line(s) passing through $(5/2, 0)$ tangent to the graph of $y = f(x)$.

Solution: For any random number a, let's first write down (in terms of a) the equation of the line tangent to the graph of $y = f(x)$ at the point $(a, 4 - a^2)$. This line has to have slope $f'(a)$, which equals $-2a$. Therefore, by point-slope form, the equation of the line is

$$y - (4 - a^2) = -2a(x - a)$$

Now, we want to find all a with the property that this line passes through the point $(5/2, 0)$. That must mean that

$$0 - (4 - a^2) = -2a(5/2 - a)$$

$$a^2 - 4 = -5a + 2a^2$$

$$0 = a^2 - 5a + 4$$

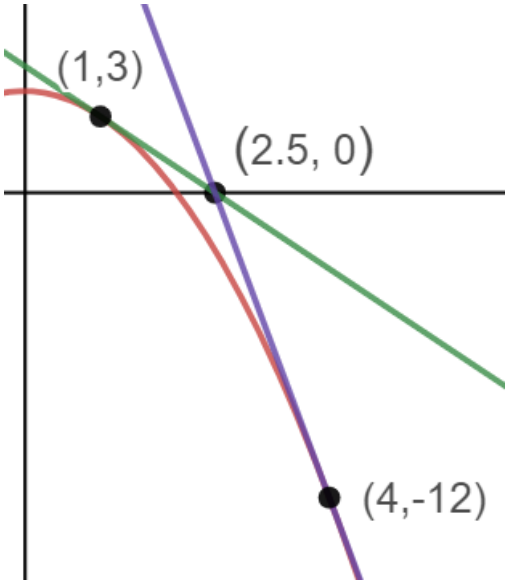
$$a = 1 \quad \text{or} \quad a = 4$$

i.e., there are two values of a which work! Plugging $a = 1$ and $a = 4$ back into the equation of the line further up, we get the two desired lines

$$y - (4 - 1^2) = -2(1)(x - 1) \quad \text{or} \quad y - (4 - 4^2) = -2(4)(x - 4)$$

Tidying these up a bit,

$$\boxed{y = -2x + 5} \quad \text{or} \quad \boxed{y = -8x + 20}$$


(1,3)
(2.5, 0)
(4,-12)